Let V denote the matrix to define the normal mode, i.e.,

$$\boldsymbol{U} = \mathrm{V}\boldsymbol{u}\,, \tag{68}$$

where $\boldsymbol{U} = (X, P_x, Y, P_y, Z, P_z)$ and $\boldsymbol{u} = (x, p_x, y, p_y, z, \delta \equiv p-1)$ are the normal and physical coordinates, respectively. The matrix V can be expressed as

$$\mathrm{V} = \mathrm{P}\mathrm{B}\mathrm{R}_6\mathrm{H}\,, \tag{69}$$

where

$$\mathrm{H} = \begin{pmatrix} \left(1 - \dfrac{\det \mathrm{H}_x}{1+a}\right)\mathrm{I} & \dfrac{\mathrm{H}_x\mathrm{J}_2\mathrm{H}_y^T\mathrm{J}_2}{1+a} & -\mathrm{H}_x \\ \dfrac{\mathrm{H}_y\mathrm{J}_2\mathrm{H}_x^T\mathrm{J}_2}{1+a} & \left(1 - \dfrac{\det \mathrm{H}_y}{1+a}\right)\mathrm{I} & -\mathrm{H}_y \\ -\mathrm{J}_2\mathrm{H}_x^T\mathrm{J}_2 & -\mathrm{J}_2\mathrm{H}_y^T\mathrm{J}_2 & a\mathrm{I} \end{pmatrix}, \tag{70}$$

$$\mathrm{R}_6 = \begin{pmatrix} \mathrm{R} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \mathrm{I} \end{pmatrix} = \begin{pmatrix} b\mathrm{I} & \mathrm{J}_2\mathrm{r}^T\mathrm{J}_2 & 0 \\ \mathrm{r} & b\mathrm{I} & 0 \\ 0 & 0 & \mathrm{I} \end{pmatrix}, \tag{71}$$

$$\mathrm{PB} = \begin{pmatrix} \mathrm{P}_x\mathrm{B}_x & 0 & 0 \\ 0 & \mathrm{P}_y\mathrm{B}_y & 0 \\ 0 & 0 & \mathrm{P}_z\mathrm{B}_z \end{pmatrix}, \tag{72}$$

with

$$a^2 + \det \mathrm{H}_x + \det \mathrm{H}_y = 1\,, \tag{73}$$

$$b^2 + \det \mathrm{R} = 1\,. \tag{74}$$

Symbols I,J_2, $\mathrm{H}_{x,y}$, r, $\mathrm{B}_{x,y,z}$, $\mathrm{P}_{x,y,z}$ above are 2 by 2 matrices:

$$\mathrm{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{75}$$

$$\mathrm{J}_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{76}$$

$$\mathrm{r} \equiv \begin{pmatrix} \mathtt{R1} & \mathtt{R2} \\ \mathtt{R3} & \mathtt{R4} \end{pmatrix}, \tag{77}$$

$$\mathrm{B}_{x,y} \equiv \begin{pmatrix} \dfrac{1}{\sqrt{\beta_{x,y}}} & 0 \\ \dfrac{\alpha_{x,y}}{\sqrt{\beta_{x,y}}} & \sqrt{\beta_{x,y}} \end{pmatrix}, \tag{78}$$

$$\mathrm{P}_{x,y,z} \equiv \begin{pmatrix} \cos\psi_{x,y,z} & \sin\psi_{x,y,z} \\ -\sin\psi_{x,y,z} & \cos\psi_{x,y,z} \end{pmatrix}. \tag{79}$$

Matrices $\mathrm{H}_{x,y}$ define dispersions as

$$\begin{pmatrix} \mathtt{ZX} & \mathtt{EX} \\ \mathtt{ZPX} & \mathtt{EPX} \\ \mathtt{ZY} & \mathtt{EY} \\ \mathtt{ZPY} & \mathtt{EPY} \end{pmatrix} \equiv \mathrm{R}\begin{pmatrix} \mathrm{H}_x \\ \mathrm{H}_y \end{pmatrix}. \tag{80}$$