We assume the focusing is uniform and $K = B'/(B\rho) > 0$, which can be always obtained by an appropriate rotation in the x-y plane. For a uniform longitudinal field $B_z = B/(B\rho)$ with the length of the section ℓ, the solution of H_2 for $1/\rho = 0$ is written as:

$$
\begin{aligned}
x &= x_0 + \Delta x\,, &(20)\\
p_x &= p_{x0} + p u_2 + B_z\left(v_2 - \frac{\Delta y}{2}\right)\,, &(21)\\
y &= y_0 + \Delta y\,, &(22)\\
p_y &= p_{y0} + p w_+ v_1 + B_z\left(-\frac{u_1}{w_+} + \frac{\Delta x}{2}\right)\,, &(23)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta x &= \frac{u_1}{w_1} + \frac{v_1 Bz}{p w_2}\,, &(24)\\
\Delta y &= \frac{u_2 B_z}{p w_1 w_+} + \frac{w_+ v_2}{w_2}\,, &(25)\\
u_1 &= a w_1(\cos\phi_1 - 1) + b\sin\phi_1\,, &(26)\\
u_2 &= -a w_1 \sin\phi_1 + b(\cos\phi_1 - 1)\,, &(27)\\
v_1 &= c w_2(\cosh\phi_2 - 1) + d\sinh\phi_2\,, &(28)\\
v_2 &= c w_2\sinh\phi_2 + d(\cosh\phi_2 - 1)\,, &(29)
\end{aligned}
$$

with

$$
\begin{aligned}
a &= U \times \left(w_2 w_+ x_0 - \frac{B_z}{p^2} p_{ym}\right)\,, &(30)\\
b &= \frac{w_1 U}{p} \times (w_+ p_{xm} - B_z w_2 y_0)\,, &(31)\\
c &= \frac{U}{p} \times \left(\frac{w_1 B_z}{w_+} x_0 + p_{ym}\right)\,, &(32)\\
d &= U \times w_2\left(-\frac{B_z}{p^2 w_+} p_{xm} + w_1 y_0\right)\,, &(33)\\
p_{xm} &= p_{x0} + \frac{B_z}{2} y_0\,, &(34)\\
p_{ym} &= p_{y0} - \frac{B_z}{2} x_0\,. &(35)
\end{aligned}
$$

The parameters are:

$$
\begin{aligned}
\phi_1 &= w_1 \ell\,, &(36)\\
\phi_2 &= w_2 \ell\,, &(37)\\
w_1 &= \sqrt{\frac{(B_z/p)^2 + V}{2}}\,, &(38)\\
w_2 &= \frac{K}{p w_1}\,, &(39)\\
w_+ &= w_1 + w_2\,, &(40)\\
U &= \frac{1}{V}\,, &(41)\\
V &= \sqrt{(B_z/p)^4 + 4(K/p)^2} = w_1^2 + w_2^2\,. &(42)
\end{aligned}
$$

The subscript 0 above denotes the initial value.
The second order Hamiltonian H_2 can be rewritten to

$$
H_{2u} = -p - i w_1 u p_u - w_2 v p_v \qquad (43)
$$

in terms of a *complex* normal coordinate:

$$
\begin{pmatrix} u \\ p_u \\ v \\ p_v \end{pmatrix} = \sqrt{U}
\begin{pmatrix}
\frac{w_1+w_2}{2} & -\frac{i}{p} & -\frac{\sqrt{w_1^2-w_2^2}}{2} & \frac{1}{p}\sqrt{\frac{w_1-w_2}{w_1+w_2}} \\
-\frac{ip}{4}(w_1+w_2)^2 & \frac{w_1+w_2}{2} & -\frac{p(w_1-w_2)}{4}\sqrt{w_1^2-w_2^2} & -\frac{w_1-w_2}{2}\sqrt{\frac{w_1-w_2}{w_1+w_2}} \\
-\frac{\sqrt{w_1^2-w_2^2}}{2} & \frac{1}{p}\sqrt{\frac{w_1-w_2}{w1+w2}} & \frac{w_1+w_2}{2} & \frac{1}{p} \\
-\frac{p(w_1-w_2)}{4}\sqrt{w_1^2-w_2^2} & -i\frac{w_1-w_2}{2}\sqrt{\frac{w_1-w_2}{w_1+w_2}} & -\frac{ip}{4}(w_1+w_2)^2 & \frac{w_1+w_2}{2}
\end{pmatrix}
\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}. \qquad (44)
$$

Note that H_{2u} is real. Thus the transformation of the longitudinal coordinate is obtained as

$$
\begin{aligned}
z &= z_0 + \left(-i u p_u \frac{\partial w_1}{\partial p} - v p_v \frac{\partial w_2}{\partial p} + \Delta v\right)\ell &(45)\\
&= z_0 + \frac{U}{p}\left(i w_1^3 u_0 p_{u0} + w_2^3 v_0 p_{v0} + \Delta v\right)\ell\,, &(46)
\end{aligned}
$$

using $u p_u = u_0 p_{u0}$ and $v p_v = v_0 p_{v0}$.